

# HOMOCLINIC INTERSECTIONS FOR GEODESIC FLOWS ON CONVEX SPHERES

ZHIHONG XIA AND PENGFEI ZHANG

**ABSTRACT.** In this paper we study some generic properties of the geodesic flows on a convex sphere. We prove that,  $C^r$  generically ( $2 \leq r \leq \infty$ ), every hyperbolic closed geodesic admits some transversal homoclinic orbits.

## 1. INTRODUCTION

Let  $f : M \rightarrow M$  be a diffeomorphism on a closed manifold  $M$ ,  $p$  be *hyperbolic* periodic point of period  $n$ , and  $W^{s,u}(p)$  be the stable and unstable manifolds of  $p$ . A point  $x \in W^s(p) \cap W^u(q)$  is called a *heteroclinic intersection* (a *homoclinic intersection* if  $q = p$ ). The intersection  $W^s(p) \cap W^u(q)$  at  $x$  is said to be *transverse* if  $T_x W^s(p) \oplus T_x W^u(q) = T_x M$ . Transverse homoclinic intersection plays an important role in the birth of the concept *Dynamical Systems*. The complexity caused by transverse homoclinic intersections made Poincaré once to believe such phenomenon can not exist in nature. Later in [31], Poincaré described the first geometric picture that every transverse homoclinic intersection is accumulated by infinitely many other homoclinic points, and this mechanism generates various complicated dynamical behaviors. The study of transverse homoclinic intersections was developed by Birkhoff [7] and then by Smale [36]. The geometric model, now called *Smale horseshoe*, gives a symbolic coding of the dynamics around a transverse homoclinic intersection, and paves the way for a systematic study of general dynamical systems with some hyperbolicity.

In this paper we study the existence of homoclinic intersections for the geodesic flow on convex spheres. Let  $2 \leq r \leq \infty$ ,  $\mathcal{G}^r$  be the set of  $C^r$ -smooth Riemannian metrics on  $S^2$ ,  $\mathcal{G}_+^r$  be the subset of Riemannian metrics on  $S^2$  with positive curvature. Endowed with  $C^r$  topology, both sets are Baire spaces. Given a Riemannian metric  $g$ , let  $\phi_t^g$  be the induced geodesic flow on the tangent bundle  $TS^2$ . Note that the geodesic flow preserves the length of tangent vectors. So one usually consider the restriction of  $\phi_t^g$  on the unit tangent bundle  $M_g := \{(x, v) \in TS^2 : \|v\|_{g(x)} = 1\}$ .

The dynamical properties of geodesic flows on  $S^2$  have been studied extensively for a long time. The simple topology of  $S^2$  leads one to believe that the geodesic flow on  $S^2$  shouldn't be chaotic. Very surprisingly, Donnay constructed in [14] a smooth metric on  $S^2$  whose geodesic flow is *nonuniformly hyperbolic* and hence extremely chaotic. For example the geodesic flow even has positive metric entropy. See [8] for an analytic example. Note that these chaotic metrics have negative curvature almost everywhere (except three small caps). In [22] Knieper and Weiss constructed the first example of positively curved sphere whose geodesic flow has positive topological entropy. In [15] Donnay proved that a small perturbation of the metric can make some non-transverse intersections to become transverse. Then in [12] Contreras and Paternain proved the  $C^2$ -denseness of metrics on  $S^2$  with positive topological entropy. Under an extra assumption that the curvature is everywhere positive, Knieper and Weiss [23] prove the  $C^\infty$ -denseness metrics on  $S^2$  with positive topological entropy. They also observed in [23] that the curvature assumption can be removed by using a result in [20]. The following theorem improves the characterizations of the geodesic flows on generic convex spheres:

**Theorem 1.** *There is a residual subset  $\mathcal{R}^r \subset \mathcal{G}_+^r$ , such that for each  $g \in \mathcal{R}^r$ , the geodesic flow  $\phi_t^g$  on  $M_g$  satisfies the following properties:*

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- (1) *every closed orbit is either hyperbolic or irrationally elliptic* [1, 2];
- (2) *the stable and unstable manifolds of two hyperbolic closed geodesics admit some transverse intersections whenever they intersect.*
- (3) *every hyperbolic closed geodesic admits some transverse homoclinic intersections.*

The last item in the above theorem is about the existence of homoclinic intersections for *all* hyperbolic closed geodesics. This is one of the two properties suggested by Poincaré [31]: a generic map  $f \in \text{Diff}_\mu^r(M)$  satisfies the following:

- (P1) (hyperbolic) periodic points are *dense* in the space  $M$ ;
- (P2) for every hyperbolic periodic point  $p$ ,
  - (P2a)  $W^s(p) \cap W^u(p) \setminus \{p\} \neq \emptyset$  (weak version);
  - (P2b)  $W^s(p) \cap W^u(p)$  is *dense* in  $W^s(p) \cup W^u(p)$  (strong version).

These properties are closely related to the *Closing problem* and *Connecting problem*, and have been one of the main motivations for the recent development in dynamical systems. Both parts have been proved when  $r = 1$  (see [33, 34] and [38, 40]). There are some partial results when  $r > 1$  (mainly in 2D). Pixton [30], extending a previous result of Robinson [35], proved (P2a) when  $M = S^2$ . That is, for a  $C^r$  generic area-preserving diffeomorphism on  $S^2$ , every hyperbolic periodic point  $p$  admits some homoclinic intersections. Using some topological argument in [27], Oliveira showed the generic existence of homoclinic intersections on  $\mathbb{T}^2$ . For a general surface  $S$ , there are some cases that (P2a) has been proved: [28] among those whose induced actions  $f_*$  on the first homology group  $H_1(S)$  are irreducible, and [41] among the class of Hamiltonian diffeomorphisms. Very recently, Asaoka and Irie [3] proved (P1) among Hamiltonian diffeomorphism on any surfaces.

Closing and connecting problems for geodesic flows are still open even in the  $C^1$  category (that is,  $C^2$  metrics), almost 50 years after Pugh proved the  $C^1$  Closing Lemma in 1967. The difficulty is that one can not perturb the geodesic flow directly: all perturbations have to be done through the perturbation of the metric on the manifold; but such perturbations, changes the dynamics of all geodesic passing through that region. Similar difficulty appears in the study of generic properties of dynamical billiards. Recently in [42, 43], we proved that (P2a) holds for generic convex billiards. Theorem 1 shows that (P2a) holds for geodesic flows induced by convex metrics on  $S^2$ .

## 2. PRELIMINARIES

Let  $M$  be a closed 3D manifold,  $V$  be a nonsingular vector field on  $M$ , that is,  $V(x) \neq 0$  for any  $x \in M$ , and  $\phi_t$  be the flow induced by  $V$ . A point  $x$  is said to be *periodic*, if the orbit through  $x$  is closed:  $\phi_t(x) = x$  for some time  $t > 0$ . The period of  $x$  is the minimal time  $t > 0$  such that  $\phi_t(x) = x$ . For each closed orbit  $\gamma$  one can draw a local cross-section  $\Sigma$  through  $x \in \gamma$  and define the first return map, the *Poincaré map*  $P = P_{\gamma, \Sigma}$ , from (a smaller subset of)  $\Sigma$  to itself under  $\phi_t$ . Note that  $P(x) = x$ , and the linearization  $D_x P$  acts linearly on  $T_x \Sigma$ . Let  $\text{Tr}(\gamma)$  be the *trace* of the linear action  $D_x P$ . Note that  $\text{Tr}(\gamma)$  is independent of the choices of  $\Sigma$  and of the choices of  $x \in \gamma$ . More generally, let  $\gamma^k$  be the closed orbit that travels along  $\gamma$   $k$  times. We can study the  $k$ -th return map  $P^k$  and define  $\text{Tr}(\gamma^k)$  to be the trace of the action  $D_x P^k$ .

Assume that  $P$  preserves an area-form on  $\Sigma$ . Then the closed orbit  $\gamma$  is said to be *degenerate*, if  $\text{Tr}(\gamma) = 2$ ; to be *nondegenerate* if  $\text{Tr}(\gamma) \neq 2$ . Note that each nondegenerate closed orbit is persistent under small perturbations of the vector field  $V$ . Moreover, the closed orbit  $\gamma$  is said to be *hyperbolic* if  $|\text{Tr}(\gamma)| > 2$ ; to be *elliptic* if  $|\text{Tr}(\gamma)| < 2$  and to be *parabolic* if  $|\text{Tr}(\gamma)| = 2$ . For an elliptic point  $x$ ,  $D_x P$  is similar to a rotation matrix  $R_\rho = \begin{bmatrix} \cos 2\pi\rho & -\sin 2\pi\rho \\ \sin 2\pi\rho & \cos 2\pi\rho \end{bmatrix}$  (for some  $0 < \rho < 1$ ). In this case the number  $\rho$  is also called the *rotation number* of the elliptic closed orbit  $\gamma$ . Then  $\gamma$  is said to be *irrationally elliptic*, if the rotation number  $\rho$  of  $\gamma$  is irrational.

**2.1. Geodesic flow.** Let  $X$  be a closed surface endowed with a smooth Riemannian metric  $g$ . For a point  $x \in X$  and a vector  $v \in T_x X$ , let  $\gamma_v(t) = \exp_x(tv)$  be the geodesic starting at  $x$  with initial velocity  $v$ . This induces a smooth map on the tangent bundle  $\phi_t : TX \rightarrow TX, (x, v) \mapsto (\gamma_v(t), \dot{\gamma}_v(t))$ .

Note that  $|\dot{\gamma}_v(t)| \equiv |v|$ . So we can restrict the flow  $\phi_t$  to the unit tangent bundle  $M_g = \{(x, v) \in TX : g_x(v, v) = 1\}$ . Note that the manifold  $M_g$  changes when we perturb the metric  $g$ . To avoid this problem, we can consider the abstract sphere bundle  $M = (TX \setminus \{0_X\})/\mathbb{R}_+$ , where  $0_X : X \rightarrow TX$  is the zero section and  $\mathbb{R}_+ = (0, \infty)$ . Note that there is a canonical isomorphism  $i_g : M_g \rightarrow M$ , and one can study the abstract geodesic flow  $i_g \circ \phi_t \circ i_g^{-1}$  on  $M$ . We will not distinguish these two settings.

Let  $TTX$  be the double tangent bundle of  $X$ . Note that the fiber  $T_{(x,v)}(TX)$  can be identified with  $T_x X \oplus T_x X$  via the map  $\xi \mapsto (d\pi(\xi), C(\xi))$ , where  $\pi : TX \rightarrow X$  is the natural projection and  $C : T(TX) \rightarrow TX$  is the connection map induced by the Levi-Civita connection related to the Riemannian metric  $g$ . Under this isomorphism,  $T_{(x,v)}(M_g)$  is mapped to  $T_x X \oplus v^\perp$ . Moreover, the vector field  $G : M_g \rightarrow TM_g$  generating the geodesic flow  $\phi_t$  on  $M_g$  is given by  $G(x, v) = (v, 0)$ .

**2.2. Bumpy Metric Theorem.** Let  $\phi_t$  be the geodesic flow on the unit tangent bundle of the closed surface  $(X, g)$ . A periodic orbit of the geodesic flow  $\phi$  corresponds to a closed geodesic  $\gamma$  on  $X$ , and the minimal period of the orbit, say  $T$ , corresponds the prime length of the closed geodesic  $\gamma$ . The Riemannian metric  $g$  is said to be *bumpy*, if every closed geodesic, viewed as a periodic orbit of the geodesic flow on the unit tangent bundle, is either hyperbolic or irrationally elliptic (that is, the rotation number  $\rho$  is irrational if  $\gamma$  is elliptic).

Let  $2 \leq r \leq \infty$ ,  $\mathcal{G}^r$  be the space of  $C^r$  Riemannian metrics on  $X$ . The following theorem was formulated by Abraham in [1] and proved by Anosov in [2].

**Bumpy Metric Theorem.** *The set of bumpy metrics on  $X$  is residual in  $\mathcal{G}^r$ .*

**2.3. Transversal intersections.** Donnay proved the following perturbation result in [15].

**Proposition 1.** *Let  $(X, g)$  be a closed surface and  $\gamma, \eta$  be two hyperbolic closed geodesics, such that  $W^s(\gamma)$  and  $W^u(\eta)$  admit a non-transverse intersection. Then there is a  $C^r$ -small perturbation  $\hat{g}$  of the metric  $g$ , such that  $W^s(\hat{\gamma})$  and  $W^u(\hat{\eta})$  admits a transverse intersection.*

In [15] the result is stated under a stronger assumption that two components with non-transverse intersection coincide. This coincidence is used to find an isolated location  $U \subset X$  far away from  $\gamma \cup \eta$  such that a local perturbation of the metric  $g$  on  $U$  does not change the dynamics near  $\gamma \cup \eta$  and make the intersection over  $U$  transverse. However, this isolated place exists for any non-transverse intersection and the same argument works under the assumptions in Proposition 1. Note that a transverse intersection, once created, is stable under small perturbations.

**2.4. The Birkhoff section and annulus map.** It is well known that there are infinitely many closed geodesic on any closed surface. For surfaces  $X \neq S^2$ , this is trivial since  $\pi_1(X) \neq 0$  and the curves minimizing the length/energy among the closed curves  $c : S^1 \rightarrow X$  within a given nonzero homotopy class must be geodesic. In the case  $X = S^2$  endowed with a general Riemannian metric  $g$ , the above *curve shortening process* does not work, since any closed curve can be shortened to a point curve. Instead of examining the closed curves one by one, we consider the free loop space  $\Lambda S^2$  on  $S^2$  (or more generally, the 1-cycle space, see [9]). We know that  $\pi_1(\Lambda S^2) \simeq \pi_2(S^2) \neq 0$ . Let  $\eta$  be a closed curve on  $\Lambda S^2$  such that  $[\eta] \neq 0 \in \pi_1(\Lambda S^2)$ . For each  $t$ ,  $\eta(t) \in \Lambda S^2$  is a closed curve on  $S^2$ . Let  $l(g) = \min_{[\eta] \neq 0} \max_{t \in S^1} |\eta(t)|$ . Then there exists a nontrivial closed geodesic  $\gamma_g$  on  $S^2$  of length  $l(g)$ .

This is the *minimax argument* introduced by Birkhoff [5]. Moreover, if we assume the sphere  $(S^2, g)$  is convex, then

- $l(g)$  is the minimum of the lengths of closed geodesics on  $S^2$ , since each closed geodesic  $\gamma$  can be embedded to a loop  $\eta : S^1 \rightarrow \Lambda S^2$  with  $\eta(0) = \gamma$  and  $|\eta(t)| \leq |\gamma|$  for all  $t$ ;
- any closed geodesic of length  $l(g)$  is simple, that is,  $\gamma$  has no self-intersection on  $S^2$ . For example, if  $\gamma$  is of figure eight on  $S^2$ , one can first shift  $\gamma$  along the normal direction to get a nongeodesic but shorter figure eight curve, say  $\sigma$ . Then we can embed  $\sigma$  to a path  $\eta$  with  $|\eta(t)| \leq |\sigma| < |\gamma|$ . Therefore,  $|\gamma| > l(g)$ .

See [9] for more details. Note that there may be (infinitely) many closed geodesics with the same length, and there is no canonical way to assign to each  $g$  a closed geodesic  $\gamma_g$ .

Now let's assume the sphere  $(S^2, g)$  is convex, and  $\gamma_g$  be a simple closed geodesic given above. Let  $D_\pm$  be the two components of  $S^2 \setminus \gamma_g$ , and  $A$  be the set of unit vectors  $(x, v) \in T_{\gamma_g} S^2$  pointing to the side of  $D_+$ . For each  $(\gamma_g(t_0), v) \in A$ , let  $\theta$  be the angle measured from  $\dot{\gamma}_g(t_0)$  to  $v$ . Then we can identify  $A$  with the open annulus  $\gamma_g \times (0, \pi)$ . Birkhoff proved in [6] that for any  $(x, v) \in A$  with  $0 < \theta < \pi$ ,  $\phi_{t(x,v)}(x, v) \in A$  for some continuous positive function  $t(x, v)$  on  $A$ . Let  $F_g : A \rightarrow A$  be the Poincaré map of  $\phi$  with respect to  $A$ . Note that  $F_g$  preserves the 2-form  $\omega = \sin \theta dt \wedge d\theta$  on  $A$ . The annulus  $A$  is called a Birkhoff annulus, and the Poincaré map  $F_g$  is also called the Birkhoff annulus map with respect to  $(g, \gamma_g)$ .

A classical result of Franks [17] shows that there are infinitely many closed geodesics whenever the Birkhoff annulus map can be defined everywhere on  $A$ . See [4] for the cases when the annulus map is not defined everywhere. Note that each closed geodesic corresponds to two closed orbits of the geodesic flow: one moves forward, and the other one moves backward. Each one of them corresponds to a periodic orbits of  $F_g$  on  $A$ . Clearly these two have the same dynamical property. Moreover, a closed geodesic  $\gamma$  is hyperbolic if and only if the corresponding periodic orbit is hyperbolic for  $F_g$ . Very recently, Irie proved in [21] that generically, closed geodesics are dense on the sphere  $S^2$ .

**2.5. Jacobi fields.** One of the main tools to study the geodesic flow (especially on surfaces) is Jacobi field, which represents infinitesimal variations around a geodesic. More precisely, let  $\gamma : \mathbb{R} \rightarrow X$  be a geodesic with unit speed, and  $\gamma_s : \mathbb{R} \rightarrow X$  be a smooth family of geodesics with  $\gamma_0 = \gamma$ . Then  $J(t) = \frac{d}{ds} \Big|_{s=0} \gamma_s$  defines a vector field along  $\gamma$ , the so-called Jacobi field. A Jacobi field  $J(t)$  along a geodesic  $\gamma$  satisfies the Jacobi equation:  $\ddot{J} + R(J, \dot{\gamma})\dot{\gamma} = 0$ , where  $R$  is the curvature tensor. In fact, the evolution of Jacobi fields with time is governed by the tangent of the geodesic flow. That is, let  $\gamma(t)$  be the geodesic with initial position  $(x, v)$ ,  $J(t)$  be the Jacobi field along  $\gamma$  with the initial condition  $(J(0), \dot{J}(0)) = (J_0, \dot{J}_0)$ . Under the identification  $T_{(x,v)}TX$  with  $T_x X \times T_x X$ , we have  $D_{(x,v)}\phi_t(J_0, \dot{J}_0) = (J(t), \dot{J}(t))$ , see [29].

Note that if  $J$  and  $\dot{J}$  are orthogonal to  $\dot{\gamma}$  at some time  $t = t_0$ , then they are orthogonal to  $\dot{\gamma}$  for all times  $t$ . Such Jacobi fields are called *orthogonal*. For orthogonal Jacobi fields, the Jacobi equation can be rewritten as  $\ddot{J}(t) + K(t) \cdot J(t) = 0$ , where  $K(t)$  is the Gaussian curvature at  $\gamma(t)$ . Moreover, let  $N$  be the normal vector field along  $\gamma$ , and let  $J = f(t) \cdot N$ . Then we have a scalar ODE  $\ddot{f} + K \cdot f = 0$ .

The orthogonal Jacobi fields lie in a 2D subbundle  $\mathcal{J} \subset TM_g$ , where  $\mathcal{J}_{(x,v)} = v^\perp \oplus v^\perp$ . In particular, for closed geodesic  $\gamma$  with minimal period  $T$ ,  $D_{(x,v)}\phi_T$  induces a linear action on the 2D space  $\mathcal{J}_{(x,v)}$  of orthogonal Jacobi fields, and this is exactly the linearized Poincaré map  $D_{(x,v)}F$  with respect to some transversal  $\Sigma$  with  $T_{(x,v)}\Sigma = \mathcal{J}_{(x,v)}$ . More precisely, let  $J_1(t) = f_1(t)N_{\gamma(t)}$  be the Jacobi field on  $\gamma$  with initial condition  $(f_1(0), f_1'(0)) = (1, 0)$ . Note that  $J(T)$  may be different from  $J(0)$  although  $\gamma$  is  $T$ -periodic. Then  $J_2(t) = f_2(t)N_{\gamma(t)}$  with  $f_2(t) := f_1(t) \cdot \int_0^t \frac{ds}{f_1(s)^2}$  is the Jacobi field on  $\gamma$  with initial condition  $(f_2(0), f_2'(0)) = (0, 1)$ . Note that  $f_2(t)$  is well defined at the times when  $f_1(t_0) = 0$ , since in this case we must have  $f_1'(t_0) \neq 0$  and  $\lim_{t \rightarrow t_0} f_2(t) = -\frac{1}{f_1'(t_0)}$ . Then

$$D_{(x,v)}F = D_{(x,v)}\phi_T|_{\mathcal{J}} = \begin{bmatrix} f_1(T) & f_2(T) \\ f_1'(T) & f_2'(T) \end{bmatrix}, \quad (2.1)$$

$$\text{Tr}(\gamma) = f_1(T) + f_2'(T) = f_1(T) + \frac{1}{f_1(T)} + f_1'(T) \int_0^T \frac{ds}{f_1(s)^2}. \quad (2.2)$$

### 3. PERTURBATIONS OF CLOSED GEODESICS

Let  $(X, g)$  be a closed surface,  $M_g \subset TX$  be the unit tangent bundle and  $\phi_t$  be the geodesic flow on  $M_g$ . Let  $\gamma$  be a closed geodesic on  $X$ ,  $\Sigma \subset M_g$  be a cross-section of the geodesic flow at some point

$(x, v) = (\gamma(t_0), \dot{\gamma}(t_0))$ ,  $F = F_{\gamma, \Sigma}$  be the Poincare map of the geodesic flow on  $\Sigma$ ,  $\text{Tr}(\gamma) = \text{Tr}(D_{(x,v)}F)$  be the trace of the linearized action. Note that  $\text{Tr}(\gamma)$  is independent of the choices  $x \in \gamma$  and of the choices of transversals  $\Sigma$ . Moreover,  $\text{Tr}(\gamma) = f_1(T) + f_2'(T)$ , see Eq. (2.2).

**Proposition 2.** *Let  $(X, g)$  be a closed surface and  $\gamma$  be a closed geodesic. Then there exists a  $C^r$  perturbation  $\hat{g}$  of the metric  $g$ , such that  $\gamma$  is still a closed geodesic for the new metric  $\hat{g}$ , and the trace  $\text{Tr}_{\hat{g}}(\gamma) \neq \text{Tr}_g(\gamma)$ .*

Note that the above result is a weak version of Franks' Lemma for perturbations of abstract diffeomorphisms. Franks' Lemma for geodesic flows turns out to be very difficult to prove, and may not hold on its full generality, see [10, 39]. Proposition 2 is sufficient for our need in this paper.

*Proof.* Let  $\gamma$  be a closed geodesic and  $T > 0$  be its prime period. Note that  $\gamma$  may have finitely many self-intersections on  $X$ . Such points are called the multi-points of  $\gamma$ . We fix a simple point on  $\gamma$ , say  $x_0 = \gamma(t_0)$ , in the sense that  $\gamma(t) \neq x_0$  for any  $t \in [0, T) \setminus \{t_0\}$ . Without loss of generality we assume  $t_0 = 0$ . Pick  $\epsilon > 0$  small enough such that  $\gamma([T - 2\epsilon, T])$  consists only of simple points of  $\gamma$ , and  $U$  a small tubular neighborhood of  $\gamma(T - 2\epsilon, T)$  such that  $\gamma \cap U = \gamma(T - 2\epsilon, T)$ . This  $U$  will be the support of our perturbation of the metric  $g$ .

Let  $J_1(t) = f_1(t)N_{\gamma(t)}$  be the Jacobi field on  $\gamma$  with initial condition  $(f_1(0), f_1'(0)) = (1, 0)$ . Note that  $(f_1(t), f_1'(t)) \neq (0, 0)$  for any  $t$ . In the following we first assume  $f_1(T) \neq 0$  and  $f_1'(T) \neq 0$ . The cases that  $f_1(T)f_1'(T) = 0$  will be discussed at the end of the proof.

Let  $h : [0, T] \rightarrow [0, 1]$  be a  $C^{r+2}$  small function satisfying the following conditions:

- (1)  $h(t) = 0$  for all  $0 \leq t \leq T - 2\epsilon$  and for all  $T - \epsilon \leq t \leq T$ ;
- (2)  $h(t) > 0$  for all  $T - 2\epsilon < t < T - \epsilon$ .

Let  $\hat{f}_1(t) = f_1(t) + h(t)$  and  $\hat{K}(t) = -\frac{\hat{f}_1''(t)}{\hat{f}_1(t)}$  for each  $0 \leq t \leq T$ . It is easy to see that  $\hat{K}$  is smooth,  $T$ -periodic and  $C^r$ -close to  $K$ . Moreover,  $\hat{J}(t) = \hat{f}(t)N_{\gamma(t)}$  will be a new Jacobi field with initial condition  $(\hat{f}_1(0), \hat{f}_1'(0)) = (1, 0)$  if  $\hat{K}$  describes the curvature along  $\gamma$  for some metric  $\hat{g}$ . Assuming this for a moment, we see that  $\hat{f}_1(T) = f_1(T)$ ,  $\hat{f}_1'(T) = f_1'(T)$ ,

$$\begin{aligned} \text{Tr}_{\hat{g}}(\gamma) &= \hat{f}_1(T) + \frac{1}{\hat{f}_1(T)} + \hat{f}_1'(T) \int_0^T \frac{ds}{\hat{f}_1(s)^2} \\ &= f_1(T) + \frac{1}{f_1(T)} + f_1'(T) \int_0^T \frac{ds}{(f_1(s) + h(s))^2}, \\ \text{Tr}_{\hat{g}}(\gamma) - \text{Tr}_g(\gamma) &= f_1'(T) \int_{T-2\epsilon}^{T-\epsilon} \left( \frac{1}{(f_1(s) + h(s))^2} - \frac{1}{f_1(s)^2} \right) ds. \end{aligned}$$

Therefore,  $\text{Tr}_{\hat{g}}(\gamma) \neq \text{Tr}_g(\gamma)$  and depends continuously on the function  $h$ .

To construct a metric  $\hat{g}$  such that  $\hat{K}(t)$  is the curvature along  $\gamma(t)$ , we will use the Fermi coordinate along  $\gamma$ , that is  $(t, s) \mapsto \exp_{\gamma(t)}(sN(t))$ . In this coordinate system, the metric tensor  $g$  satisfies

- (1)  $g_{11}(t, 0) = 1$ ,  $g_{12}(t, s) = g_{21}(t, s) = 0$  and  $g_{22}(t, s) = 1$ ;
- (2)  $\partial_s g_{11}(t, 0) = 0$ ,  $\Gamma_{jk}^i(t, 0) = 0$  for all  $i, j, k$ .
- (3)  $\partial_s^2 \sqrt{g_{11}}(t, 0) = -K(t)$ , or equivalently,  $\partial_s^2 g_{11}(t, 0) = -2K(t)$ .

Then the new metric tensor  $\hat{g}$  on  $U$  is given by

- (1)  $\hat{g}_{ij}(t, s) = 0$  when  $i \neq j$ ,  $\hat{g}_{22}(t, s) = 1$ ,
- (2)  $\hat{g}_{11}(t, s) = g_{11}(t, s) - k(t)b(s)s^2$ ,

where  $k(t) = \hat{K}(t) - K(t)$ , and  $b$  be a smooth bump function with  $b(0) = 1$  and a uniform  $C^r$ -norm. Then  $\hat{g}$  is  $C^r$  close to  $g$  and is identical to  $g$  on  $X \setminus U$ . Note that the curve  $(t, 0) \mapsto \gamma(t)$  is still a closed geodesic under  $\hat{g}$ , and the new curvature at  $\gamma(t)$  is  $-\frac{1}{2}\partial_s^2 \hat{g}_{11}(t, 0) = K(t) + k(t) = \hat{K}(t)$ .

Lastly, if  $f_1(T) = 0$ , then  $f'_1(T) \neq 0$ . In this case we use a two-step perturbation of the metric: the first one makes  $\hat{f}_1(T) \neq 0$  (while keeping  $\hat{f}'_1(T) \neq 0$ ), and the second one (much smaller) changes the trace  $\text{Tr}_g(\gamma)$  continuously. The detail is omitted since the perturbations are of the same type used above. If  $f'_1(T) = 0$ , then  $f_1(T) \neq 0$  and we can employ the two-step process, too. This completes the proof.  $\square$

**Remark 1.** It follows from Proposition 2 that

- if  $\gamma$  is degenerate, then after the perturbation, it is either hyperbolic or elliptic;
- if  $\gamma$  is elliptic, then we can manipulate its rotation number (irrational, Diophantine, etc).

Recall that  $\mathcal{G}^r$  is the set of  $C^r$ -smooth Riemannian metrics on  $S^2$ ,  $\mathcal{G}_+^r$  is the subset of Riemannian metrics on  $S^2$  with positive curvature. For each  $g \in \mathcal{G}_+^r$ , let  $l(g)$  be the length of the shortest closed geodesic on the sphere  $(S^2, g)$ . See Sect. 2.4 for more details.

**Proposition 3.** *There is an open and dense subset  $\mathcal{U}^r \subset \mathcal{G}_+^r$  such that*

- (1) *for each  $g \in \mathcal{U}^r$ , there is a unique closed geodesic of length  $l(g)$ ,*
- (2) *the map  $g \in \mathcal{U}^r \mapsto \gamma_g \in C^r(S^1, S^2)$  varies smoothly.*

Therefore, on an open and dense subset  $\mathcal{U}^r \subset \mathcal{G}_+^r$ , there is a canonical choice of the simple closed geodesic  $\gamma_g$ , and the induced map  $F_g$  on  $A_g = \gamma_g \times (0, \pi)$  also depends continuously on  $g$ .

*Proof.* Let  $g_0 \in \mathcal{G}_+^r$  be a Riemannian metric on  $S^2$  whose curvature  $K_0$  is strictly positive:  $a = \min K_0(x) > 0$ . Let  $b = \max K_0(x)$ . Let  $\mathcal{U} \subset \mathcal{G}_+^r$  be a small neighborhood of  $g_0$  such that for each  $g \in \mathcal{U}$ ,  $a/2 \leq K_g(x) \leq 2b$  for any  $x \in S^2$ . Consider the function  $l : g \in \mathcal{U} \rightarrow l(g)$ . Note that there exists a constant  $C \geq \pi$  such that  $l(g)^2 \leq C \cdot \text{Area}(g_0)$  for any  $g \in \mathcal{U}$  (see [13]). For any  $g_n \rightarrow g_0$  and for any closed geodesic  $\gamma_n$  on the sphere  $(S^2, g_n)$ , the limit of  $\gamma_n$  (passing to a subsequence if necessary) is a simple closed geodesic  $\gamma$  whose length  $|\gamma| \leq \lim l(g_n)$ . Therefore,  $l(g) \leq \liminf l(g_n)$ , and the function  $l : g \in \mathcal{G}_+^r \rightarrow l(g)$  is lower semi-continuous. Let  $\mathcal{R}_l^r$  be the set of continuity of the function  $l$ , which is a residual subset of  $\mathcal{G}_+^r$ .

Let  $\mathcal{R}_b^r \subset \mathcal{G}_+^r$  be the residual subset of bumpy metrics such that each closed geodesic is either hyperbolic or irrationally elliptic. Let  $g \in \mathcal{R}_l^r \cap \mathcal{R}_b^r$ . Clearly there is only a finitely many closed geodesics of length less than  $l(g) + 1$ . Label them according to their lengths as  $\gamma_1, \dots, \gamma_k, \gamma_{k+1}, \dots, \gamma_n$ , where  $|\gamma_i| = l(g)$  for each  $1 \leq i \leq k$ , and  $|\gamma_j| \geq \gamma_{k+1} > l(g)$  for each  $k < j \leq n$ . For each  $i = 2, \dots, k$ , we select a *simple* point  $x_i$  on  $\gamma_i$ . Here, a point  $x$  is ‘simple’ if the union  $\bigcup_{1 \leq j \leq n} \gamma_j$  covers  $x$  only once. Then we perturb the metric tensor  $g$  around each  $x_i$  such that all  $\gamma_i$ ,  $2 \leq i \leq k$  are longer nondegenerate geodesics for the new metric  $\hat{g}$ :  $|\gamma_i|_{\hat{g}} \geq l(g) + \epsilon$  while  $\gamma_{\hat{g}} = \gamma_1$  is unchanged. Then there exists a very small neighborhood  $\mathcal{V}$  of  $\hat{g}$ , such that for any  $\tilde{g} \in \mathcal{V}$ , the continuation  $\gamma_{\tilde{g}}$  of  $\gamma_1$  is the only geodesic of shortest length (using the nondegeneracy condition of  $\gamma_{\tilde{g}}$ ), and  $\tilde{g} \in \mathcal{V} \mapsto \gamma_{\tilde{g}}$  varies continuously. We complete the proof by letting  $g$  vary in  $\mathcal{R}_l^r \cap \mathcal{R}_b^r$ .  $\square$

Let  $A_g$  be the set of unit tangent vectors on  $\gamma_g$  pointing to the same side of  $S^2 \setminus \gamma_g$ , and  $F_g$  be the Poincare map on  $A_g$  with respect to the geodesic flow  $\phi_t^g$ . That is,  $F_g(x, v) = \phi_{t(x,v)}^g(x, v) \in A_g$ , where  $t(x, v)$  is a continuous positive function satisfying  $0 < t_1 \leq t(x, v) \leq t_2$ . Let  $n \geq 1$ ,  $P_n(F_g)$  be the set of points in  $A_g$  that is fixed by  $F_g^n$ . Note that  $P_n(F_g)$  is always a closed (may not be compact) subset of  $A_g$ , since the return function  $t(\cdot)$  is bounded and bounded away from zero.

Let  $(x, v) \in P_n(F_g)$ . Then  $(x, v)$  is said to be *nondegenerate under  $F_g^n$*  if  $\text{Tr}(D_{(x,v)} F_g^n) \neq 2$ . Note that the minimal period  $m$  of  $(x, v)$  under  $F_g$  may be smaller than  $n$ . In this case, let  $k = \frac{n}{m}$ ,  $\gamma$  be the closed geodesic with initial condition  $(x, v)$ , and  $\gamma^k$  be the geodesic that travels along the geodesic  $\gamma$   $k$  times. Then the above nondegeneracy condition is equivalent to that  $\text{Tr}(\gamma^k) \neq 2$ , where  $k = \frac{n}{m}$ .

Let  $\mathcal{U}^r \subset \mathcal{G}_+^r$  be the open and dense subset given by Proposition 3,  $\mathcal{U}_n^r \subset \mathcal{U}^r$  be the subset of Riemannian metrics such that  $\text{Tr}(\gamma_g^n) \neq 2$  and each periodic point in  $P_n(F_g)$  is nondegenerate under  $F_g^n$ . The following Proposition 4 can be viewed as the reformulation of Bumpy Metric Theorem in

terms of its Birkhoff annulus maps. This map version formulation is the key to prove the existence of homoclinic intersections in Sect. 4.

**Proposition 4.** *Let  $n \geq 1$ , and  $\mathcal{U}_n^r$  be the subset given above. Then the following statements hold:*

- (1) *for each  $g \in \mathcal{U}_n^r$ ,  $P_n(F_g)$  is a finite subset,*
- (2) *the map  $g \in \mathcal{U}_n^r \mapsto P_n(F_g)$  varies continuously.*

*Proof.* (1). Let  $g \in \mathcal{U}_n^r$ . Suppose on the contrary that there exist infinitely many periodic points in  $P_n(F_g)$ . Pick a sequence  $(x_k, v_k)$  that converges to some point  $(x_*, v_*)$ .

- 1a).  $v_* \neq \pm \dot{\gamma}(t_0)$ : then  $(x_*, v_*) \in P_n(F_g)$ . Here we use the fact that the return time function  $t(x, v)$  is bounded. This point  $(x_*, v_*)$  must be degenerate under  $F_g^n$ , since the vector  $\lim_{x_k \rightarrow \infty} \frac{(x_k, v_k) - (x_*, v_*)}{\|(x_k, v_k) - (x_*, v_*)\|}$  is invariant under  $DF_g^n$ . This contradicts the choice of  $g \in \mathcal{U}_n^r$ .
- 1b).  $v_* = \pm \dot{\gamma}_g(t_0)$ : the geodesic  $\gamma_k(t) = \exp_{x_k}(tv_k)$  converges to the geodesic  $\gamma_g$ , which implies that  $\gamma_g^n$ , the  $n$ -iterate of  $\gamma_g$ , is degenerate under the return map  $\phi_{nT}$ . This also contradicts the choice of  $g \in \mathcal{U}_n^r$ .

(2). Note that the nondegenerate closed geodesics persist under perturbations, and the nondegeneracy under  $F_g^n$  condition is an open condition. Therefore  $g \in \mathcal{U}_n^r \mapsto P_n(F_g)$  is lower semi-continuous. The upper semi-continuity follows from the continuous dependence of  $\gamma_g$  and  $F_g$  on  $g \in \mathcal{U}_n^r \subset \mathcal{U}^r$ , see Proposition 3. Putting them together, we prove the continuity of  $P_n(F_g)$ .  $\square$

**Proposition 5.** *The set  $\mathcal{U}_n^r$  contains an open and dense subset of  $\mathcal{G}_+^r$ .*

*Proof.* The openness of  $\mathcal{U}_n^r$  follows from the finiteness and continuity dependence of  $P_n(F_g)$  in Proposition 4, while the denseness follows from Bumpy Metric Theorem.  $\square$

Let  $\gamma$  be a hyperbolic closed geodesic,  $W^s(\gamma) \setminus \gamma$  has two components  $W_\pm^s(\gamma)$ , so is  $W^u(\gamma) \setminus \gamma$ .

**Definition 1.** Let  $P_n(F_g) = \{(x_i, v_i) : 1 \leq i \leq k\}$  for some  $k = k(n)$ ,  $\gamma_i$  be the closed geodesic on  $S^2$  with initial condition  $(x_i, v_i)$ , To include the special closed geodesic  $\gamma_g$ , we re-label it by  $\gamma_0$ . Putting them together, we denote by  $\Gamma_n(g) = \{\gamma_i : 0 \leq i \leq k\}$ .

Let  $\mathcal{V}_n^r \subset \mathcal{U}_n^r$  be the subset of metrics  $g$  such that for any pair of stable and unstable components of two hyperbolic closed geodesics in  $\Gamma_n(g)$ , either they do not intersect, or they admit some transverse intersections.

**Proposition 6.** *The set  $\mathcal{V}_n^r$  contains an open and dense subset of  $\mathcal{G}_+^r$ .*

*Proof.* Let  $g_0 \in \mathcal{U}_n^r$ ,  $P_n(F_{g_0})$  be the set of points fixed by  $F_{g_0}^n$ . We can label them by  $\{(x_1, v_1), \dots, (x_k, v_k)\}$ , all being nondegenerate under  $F_{g_0}^n$ . According to Proposition 4, we can find a small neighborhood  $\mathcal{V}$  of  $g_0$  such that  $P_n(F_g) = \{(x_1(g), v_1(g)), \dots, (x_k(g), v_k(g))\}$ , where  $(x_i(g), v_i(g))$  is the continuation of  $(x_i, v_i)$ ,  $1 \leq i \leq k$ . Let  $\gamma_i$  be the closed geodesic on  $(S^2, g_0)$  with initial condition  $(x_i, v_i)$ , and  $\gamma_i(g)$  be the closed geodesic on  $(S^2, g)$  with initial condition  $(x_i(g), v_i(g))$ . To include the special closed geodesic  $\gamma_g$ , we re-label it by  $\gamma_0(g)$ . Putting them together as in Definition 1, we denote by  $\Gamma_n(g)$ .

Let  $i, j \in \{0, \dots, k\}$  be two indices such that  $\gamma_i$  and  $\gamma_j$  are hyperbolic, and  $\alpha, \beta \in \{+, -\}$  indicate the components of the stable and unstable manifolds we could pick. Note that we have added the index 0 corresponding to the geodesic  $\gamma_g$ . Let  $\mathcal{V}_{ij\alpha\beta}$  be the subset of metrics  $g$  in  $\mathcal{V}$  such that either  $W_\alpha^s(\gamma_i(g)) \cap W_\beta^u(\gamma_j(g)) = \emptyset$ , or  $W_\alpha^s(\gamma_i(g))$  and  $W_\beta^u(\gamma_j(g))$  admit some transverse intersections. It suffices to show that  $\mathcal{V}_{ij\alpha\beta}$  contains an open and dense subset in  $\mathcal{V}$ .

We partition  $\mathcal{V}$  into two parts  $\mathcal{V}_1 \cup \mathcal{V}_2$ , where

- (1)  $g \in \mathcal{V}_1$  if  $W_\alpha^s(\gamma_i(\hat{g})) \cap W_\beta^u(\gamma_j(\hat{g})) = \emptyset$  for any  $\hat{g}$  sufficiently close to  $g$ ;
- (2)  $g \in \mathcal{V}_2$  if  $W_\alpha^s(\gamma_i(g_l)) \cap W_\beta^u(\gamma_j(g_l)) \neq \emptyset$  for some  $g_l \rightarrow g$ .

It is clear that  $\mathcal{V}_1$  is open (may be empty) and  $\mathcal{V}_1 \subset \mathcal{V}_{ij\alpha\beta}$ . For each  $g \in \mathcal{V}_2$ , let  $g_l$  be a sequence given as above. If  $W_\alpha^s(\gamma_i(g_l))$  and  $W_\beta^u(\gamma_j(g_l))$  admits a transverse intersection for infinitely many  $l$ , then there exists a small neighborhood of  $g_l$  that is contained in  $\mathcal{V}_{ij\alpha\beta}$ . If  $W_\alpha^s(\gamma_i(g_l))$  and  $W_\beta^u(\gamma_j(g_l))$  intersect non-transversely, we can make a perturbation of  $g_l$  such that  $W_\alpha^s(\gamma_i(\hat{g}_l))$  and  $W_\beta^u(\gamma_j(\hat{g}_l))$  admits a transverse intersection. Then we use this new sequence  $\hat{g}_l$ . This completes the proof.  $\square$

Let  $\mathcal{R}_{KS}^r = \bigcap_{n \geq 1} \mathcal{V}_n^r$ , which contains a residual subset of  $\mathcal{G}_+^r$ .

**Theorem 2.** *There is a residual subset  $\mathcal{R}_{KS}^r$  of  $\mathcal{G}_+^r$ , such that for any  $g \in \mathcal{R}_{KS}^r$ , the geodesic flow  $\phi_t$  on the unit tangent bundle  $M_g \subset TS^2$  satisfies:*

- (1) *every closed geodesic is either hyperbolic or irrationally elliptic.*
- (2) *if  $\gamma$  and  $\eta$  are hyperbolic, then for any component of  $W_\pm^s(\gamma)$  and of  $W_\pm^u(\eta)$ , either they do not intersect, or they admit some transverse intersection.*

Let  $\mathcal{V}_n^\infty = \left( \bigcup_{r \geq 2} \mathcal{V}_n^r \right) \cap \mathcal{G}_+^\infty$ , which contains an open and dense subset of  $\mathcal{G}_+^\infty$ . Let  $\mathcal{R}_{KS}^\infty = \bigcap_{n \geq 1} \mathcal{V}_n^\infty$ , which contains a residual subset of  $\mathcal{G}_+^\infty$ . Therefore, the conclusions of Theorem 2 also hold for  $r = \infty$ .

**Remark 2.** The above theorem does not claim that the two components  $W_\pm^s(\gamma)$  and  $W_\pm^u(\eta)$  are transverse, since we do not try to remove all non-transverse intersections. More crucially, the above theorem does not specify when the two components  $W_\pm^s(\gamma)$  and  $W_\pm^u(\eta)$  can have nontrivial intersection. In the next section we will show the generic existence of homoclinic intersections for each hyperbolic closed geodesic.

#### 4. HOMOCLINIC INTERSECTIONS FOR HYPERBOLIC CLOSED GEODESICS

In this section we study the existence of homoclinic intersections of hyperbolic closed geodesics of the geodesic flow on the unit tangent bundle of the two-sphere. More precisely, let  $\mathcal{G}_+^r$  be the set of Riemannian metrics on  $S^2$  with positive Gaussian curvature,  $\mathcal{U}^r$  be the open and dense subset given by Proposition 3, such that for each  $g \in \mathcal{U}^r$ , there is exactly one simple closed geodesic  $\gamma_g$  of shortest length. Let  $A_g \subset S_g$  be the Birkhoff annulus and  $F_g$  be the Poincare map of the geodesic flow with respect to  $A_g$ . Let  $P_n(F_g)$  be the set of points fixed by  $F_g^n$ , and  $\Gamma_n(g)$  be the set of closed geodesics corresponding to  $P_n(F_g)$  with one additional  $\gamma_g$  (see Definition 1). Our main result in this section is the following.

**Proposition 7.** *There is an open and dense subset  $\mathcal{W}_n^r \subset \mathcal{G}_+^r$  such that for each  $g \in \mathcal{W}_n^r$ , for each hyperbolic closed geodesic  $\gamma \in \Gamma_n(g)$ , there exist some transverse homoclinic intersections for  $\gamma$ .*

It suffices to show such  $\mathcal{W}_n^r$  is open and dense in  $\mathcal{V}_n^r$  (see Proposition 6 and the paragraph above it for the definition of the subset  $\mathcal{V}_n^r$ ). Note that  $P_n(F_g)$  and  $\Gamma_n(g)$  are finite and vary continuously over  $g \in \mathcal{V}_n^r$ . Moreover, transverse homoclinic intersections, once created, persist under small perturbations. Therefore the set  $\mathcal{W}_n^r$  is open in  $\mathcal{V}_n^r$ . So it suffices to prove the denseness of  $\mathcal{W}_n^r$  in  $\mathcal{V}_n^r$ .

**4.1. Nonlinear stability of elliptic periodic points.** Let  $f$  be a  $C^\infty$  diffeomorphism on a surface  $S$  which preserves a smooth measure. Let  $p$  be an elliptic fixed point of  $f$ . Recall that an elliptic fixed point is also said to be *linearly stable*. Then  $p$  is said to be *nonlinearly stable*, if there exists a sequence of nesting closed disks  $D_n$ ,  $n \geq 1$  such that  $p \in D_{n+1} \subset D_n^\circ$ ,  $\bigcap_n D_n = \{p\}$  and  $f|_{\partial D_n}$  is transitive.

Recall that a real number  $\rho$  is said to be Diophantine, if there exist two positive numbers  $c, \tau$  such that

$$\left| \rho - \frac{m}{n} \right| \geq \frac{c}{|n|^{2+\tau}}, \text{ for all rational numbers } \frac{m}{n}. \quad (4.1)$$

Then an elliptic fixed point  $p$  of  $f$  is said to have Diophantine rotation number, if the tangent map  $D_p f$  is similar to a rotation  $R_\rho = \begin{bmatrix} \cos 2\pi\rho & -\sin 2\pi\rho \\ \sin 2\pi\rho & \cos 2\pi\rho \end{bmatrix}$  for some Diophantine  $\rho$ .



The following is the so called Herman's *Last Geometric Theorem*, which states that an elliptic fixed point with Diophantine rotation number is nonlinearly stable. See [16] for the history and a complete proof of this theorem.

**Herman's Last Geometric Theorem.** Let  $f \in \text{Diff}_\mu^\infty(S)$  and  $p$  be an elliptic fixed point of  $f$  with rotation number  $\rho$ . If  $\rho$  is Diophantine, then  $p$  is nonlinearly stable.

By taking a cross-section and consider the Poincare map, we see that the same result holds for elliptic closed geodesics of the geodesic flows on  $(S^2, g)$ .

We will use Herman's LGT to prove the denseness of  $\mathcal{W}_n^r$  in  $\mathcal{V}_n^r$ .

**Proposition 8.** *There is a  $C^r$  dense subset  $\mathcal{D}_n \subset \mathcal{V}_n^r \cap \mathcal{G}_+^\infty$  such that the following hold for each  $g \in \mathcal{D}_n$ :*

- (1) *every elliptic periodic point in  $P_n(F_g)$  is nonlinearly stable;*
- (2) *the shortest geodesic  $\gamma_g$  is either hyperbolic or nonlinearly stable.*

*Proof.* Let  $g \in \mathcal{V}_n^r$ ,  $\gamma_g$  be the shortest simple closed geodesic and  $F_g$  be the Poincare map of the geodesic flow  $\phi_t$  on the Birkhoff annulus  $A_g = \gamma_g \times (0, \pi)$ . Let  $P_n(F_g)$  be the set of points in  $A_g$  fixed by  $F_g^n$ , and label them as  $\{(x_i, v_i)\}_{i=1}^n$ . Let  $\gamma_i$  be the closed geodesic with initial condition  $(x_i, v_i)$  for each  $i = 1, \dots, n$  and let  $\gamma_0 = \gamma_g$ . Clearly they are all nondegenerate. For each  $j = 0, \dots, n$ , we pick a simple point  $x_j \in \gamma_j$  in the sense that the union  $\bigcup_{i=0}^n \gamma_i$  covers  $x_j$  only once. Then we perturb the metric  $g$  in a small neighborhood  $U_j$  of  $x_j$ , say the new metric  $\hat{g}$ , such that every elliptic closed geodesic among those  $\{\gamma_i(\hat{g}) : 0 \leq i \leq n\}$  has Diophantine rotation number. Then Herman's LGT guarantees that all the elliptic closed geodesics among  $\{\gamma_i(\hat{g}) : 0 \leq i \leq n\}$  are nonlinearly stable, and hence  $\hat{g} \in \mathcal{D}_n$ . Therefore,  $\mathcal{D}_n$  is dense in  $\mathcal{V}_n^r$ . This completes the proof.  $\square$

After we are done with the elliptic ones, let's move on to study the hyperbolic periodic points in  $P_n(F_g)$ . Although each point  $(x, v) \in P_n(F_g)$  is fixed by  $F_g^n$ , the two branches of the stable (and unstable) manifolds may be switched by  $F_g^n$ . Note that  $F_g^{2n}$  does fix the branches of the invariant manifolds of hyperbolic periodic points in  $P_n(F_g)$  (here we emphasize the mismatch of the indices is not a misprint). Let  $\mathcal{D}_n$ ,  $n \geq 1$  be the sequence of dense subsets of  $\mathcal{V}_n^r$  given by Proposition 8. The following lemma is proved by applying the *Prime End Compactification* method developed by Mather in [25, 26], since the topology of the manifold  $A_g$  is very simple. See also [27, 18].

**Lemma 1.** *Let  $g \in \mathcal{D}_{2n}$ . Then every hyperbolic periodic point in  $P_n(F_g)$  admits some transverse homoclinic intersections.*

*Proof.* Let  $g \in \mathcal{D}_{2n}$ ,  $\gamma_g$  be the shortest simple closed geodesic and  $F_g$  be the Poincare map of the geodesic flow  $\phi_t$  on the Birkhoff annulus  $A_g = \gamma_g \times (0, \pi)$ . For simplicity we denote  $f = F_g^{2n}$ . Note that all elliptic fixed points of  $f$  are nonlinearly stable, and any two branches of the stable and unstable manifolds of the hyperbolic fixed points are either disjoint or admit some transverse intersections.

Let  $P_n(F_g)$  be the set of points in  $A_g$  fixed by  $F_g^n$ , and  $(x, v) \in P_n(F_g)$  be a hyperbolic periodic point. Then  $(x, v)$  is a hyperbolic fixed point of  $f$ , and all four branches of the stable and unstable manifolds are also fixed by  $f$ . Then Mather's theorem [25] implies that all four branches  $W_\pm^{s,u}((x, v), f)$  have the same closure:  $\overline{W_\alpha^\sigma((x, v), f)} = E$  for each  $\sigma \in \{s, u\}$  and each  $\alpha \in \{+, -\}$ . Then we can construct a simple closed curve  $\gamma^s \subset A_g$  by closing the gate between a stable branch, say  $W_+^s(x, v) \subset A_g$ , with the fixed point  $(x, v) \in A_g$ . Suppose this gate lies in the quadrant between  $W_+^u(x, v)$  and  $W_-^s(x, v)$ . Then  $W_+^u(x, v)$  also enter this quadrant, since they have the same closure. We construct a second simple closed curve  $\gamma^u \subset A_g$  by closing the gate between  $W_+^u(x, v)$  with the fixed point  $(x, v)$ . These two simple closed curves  $\gamma^{s,u}$  intersect transversely at the fixed point  $(x, v)$ , and the algebraic sum of the intersections of two simple closed curves on an annulus must be zero. Therefore, the two curves must intersect somewhere else, and that intersection is a homoclinic intersection. See [42] for more details.  $\square$

Comparing the set  $P_n(F_g)$  with  $\Gamma_n(g)$ , we see that there is only one geodesic left from Lemma 1:  $\gamma_g$  for each  $g \in \mathcal{D}_{2n}$ .

**Remark 3.** Poincare [32] conjectured that there exists a non-hyperbolic simple closed geodesic on each convex surface. If this were true, one may try to argue that  $\gamma_g$  is non-hyperbolic. However, Grjuntal' [19] constructed an open set of convex metrics on  $S^2$  such that every simple closed geodesic is hyperbolic. Elliptic closed geodesic does exist  $C^2$  densely [11], just that it may not be simple.

Let's return to the proof of Proposition 7. If  $\gamma_g$  is elliptic, then we are done. If  $\gamma_g$  is hyperbolic, we will prove that, the choice of  $g \in \mathcal{D}_{2n}$  automatically implies the existence of homoclinic intersections. Note that no perturbation is needed in this step.

**Lemma 2.** *Let  $g \in \mathcal{D}_{2n}$  such that  $\gamma_g$  is a hyperbolic closed geodesic. Then  $\gamma_g$  admits some transverse homoclinic intersections.*

*Proof.* Let  $\eta_g$  be another simple closed geodesic given by Lyusternik-Shnirel'man theorem [24, 37], and  $\hat{F}_g$  be the new Poincare map induced on the new Birkhoff annulus  $\hat{A} = A_{\eta_g}$ . Pick  $y_0 \in \gamma_g \cap \eta_g$  and  $(y_0, u_0)$  be the point in  $\hat{A}$  that generates  $\gamma_g$ . For the second iterate  $\hat{F}_g^2$ , the following hold:

- (1) every fixed point  $(y, u)$  of  $\hat{F}_g^2$  is also fixed by  $F_g^2$ , since it corresponds to a closed geodesic  $\gamma$  that wraps around  $S^2$  at most two times;
- (2) every elliptic fixed point of  $\hat{F}_g^2$  is nonlinearly stable, since the corresponding closed geodesic is (see Proposition 8);
- (3) any two branches of the stable and unstable manifolds of two hyperbolic fixed points of  $\hat{F}_g^2$  either don't intersect or admit some transverse intersections, since the corresponding components of the closed geodesics are (see Proposition 6);
- (4)  $(y_0, u_0)$  is a hyperbolic fixed point of  $\hat{F}_g$  and all four branches are fixed by  $f$  (since  $\gamma_g$  is a simple closed geodesic).

Then the same argument given in Lemma 1 shows the existence of transverse intersections between the stable and unstable manifolds of the fixed point  $(y_0, u_0)$  for any  $g \in \mathcal{D}_{2n}$ . This completes the proof.  $\square$

*Proof of Proposition 7.* Combining Lemma 1 and 2, we see that for any  $g \in \mathcal{D}_{2n}$ , there exist transverse homoclinic intersections for any closed geodesic in  $\Gamma_n(g)$ . This implies that  $\mathcal{D}_{2n} \subset \mathcal{W}_n^r$ . Therefore  $\mathcal{W}_n^r$  must be dense in  $\mathcal{V}_n^r$ , since  $\mathcal{D}_{2n}$  is dense in  $\mathcal{V}_{2n}^r$ , which is also dense in  $\mathcal{G}_+^r$ . This completes the proof of Proposition 7.  $\square$

*Proof of Theorem 1.* Let  $\mathcal{W}_n^r$  be the open and dense subset of  $\mathcal{G}_+^r$  given by Proposition 7. Then the set  $\mathcal{R}_h^r = \bigcap_{n \geq 1} \mathcal{W}_n^r$  contains a residual subset of  $\mathcal{G}_+^r$ . Let  $g \in \mathcal{R}_h^r$ . Then each hyperbolic closed geodesic of the geodesic flow on the unit tangent bundle  $M_g$  admits some transverse homoclinic intersections. Combining with Theorem 2, we complete the proof of Theorem 1.  $\square$

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DEPARTMENT OF MATHEMATICS, SOUTH UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA SHENZHEN, GUANGDONG, CHINA 518055; PERMANENT ADDRESS: DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208.

*E-mail address:* `xiazh@sustc.edu.cn`, `xia@math.northwestern.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSISSIPPI, OXFORD, MS 38677

*E-mail address:* `pzhang2@olemiss.edu`